ON THE POSITIONS OF THE IMAGINARY POINTS OF INFLEXION AND CRITIC CENTERS OF A REAL CUBIC.

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1. **Introduction.** In the extensive study of the configuration formed by the points of inflexion of a real cubic, it appears that no one has considered the possible positions of the six imaginary points of the group when the three real points are fixed. This is worthy of consideration for these two sets of points are so related that, while the three collinear real points of inflexion impose only five conditions and hence determine a fourfold infinite system of cubics in a plane, not one of the six points can be chosen arbitrarily. The following gives a construction for such a set of six points when the three real points are taken arbitrarily on a line; and by a generalization accounts for all such possible sets of six points.

The construction for the six imaginary points of inflexion also serves to show the positions of the twelve critic centers for the non-singular real cubic.

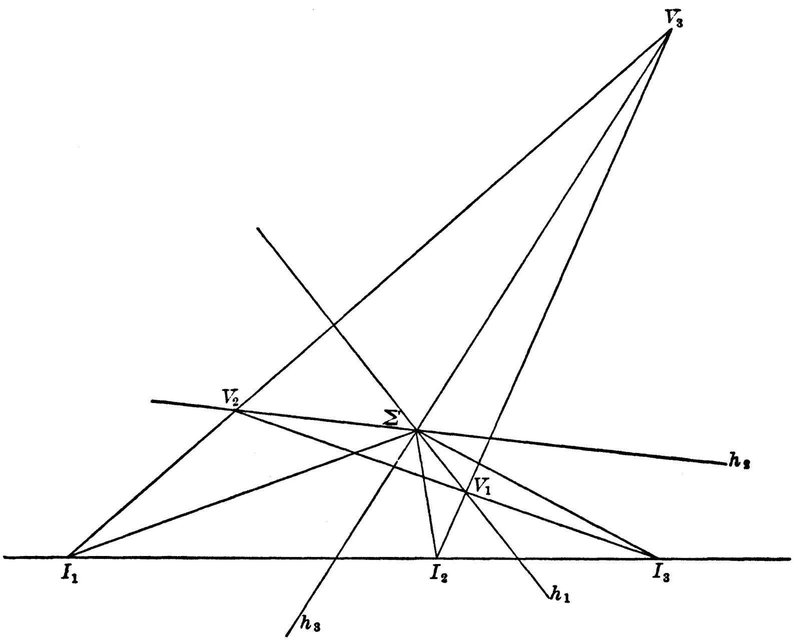

Fig. 1.

2. **Construction.** Let any three real points I_1, I_2, I_3 on an arbitrary real line be taken as points of inflexion for a real cubic. Let Σ be any

other real point. Join Σ to the points I_i and construct the fourth harmonic to each one of these lines with respect to the other two. Denote the fourth harmonic to the line through I_1 by h_1, and similarly for I_2 and I_3. Through any one of the three points, say I_1, draw an arbitrary real line intersecting h_2 and h_3 in V_2 and V_3. Draw the lines I_2V_3, I_3V_2 intersecting in V_1 on h_1. The projections upon the sides of the triangle V_1, V_2, V_3, through Σ, of the two points equianharmonic to the three points I, are six imaginary points which together with I_i form an inflexional group for a real cubic.

3. **Analytical proof of the construction.** Let $I_1(0, 1, -1)$, $I_2(-1, 0, 1)$, $I_3(1, -1, 0)$ be the three collinear points and $\Sigma(1, 1, 1)$ the arbitrary point of the plane. Then the lines joining Σ to I_i are

$$-2x + y + z = 0, \qquad x - 2y + z = 0, \qquad x + y - 2z = 0;$$

and the lines h_i are

$$h_1 : y - z = 0, \qquad h_2 : z - x = 0, \qquad h_3 : x - y = 0.$$

An arbitrary line through I_1 is $\alpha x + y + z = 0$, where α is an undetermined real number. This line cuts h_2 and h_3 in $V_2(-1, \alpha + 1, -1)$, $V_3(-1, -1, \alpha + 1)$, respectively. The lines I_2V_3, I_3V_2 have equations

$$x + \alpha y + z = 0, \qquad x + y + \alpha z = 0$$

and intersect on h_1 in $V_1(\alpha + 1, -1, -1)$.

The two points equianharmonic to I_1, I_2, I_3 are $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$, where $\omega^3 = 1$; and the lines joining these to Σ are

$$x + \omega y + \omega^2 z = 0, \qquad x + \omega^2 y + \omega z = 0.$$

These two lines intersect the sides of the triangle $V_1\, V_2\, V_3$ in

$$\begin{array}{ll}
(\omega^2 - \omega,\ 1 - \alpha\omega^2,\ \alpha\omega - 1), & (\omega - \omega^2,\ 1 - \alpha\omega,\ \alpha\omega^2 - 1); \\
(\alpha\omega - 1,\ \omega^2 - \omega,\ \alpha\omega^2 - 1), & (\alpha\omega^2 - 1,\ \omega - \omega^2,\ 1 - \alpha\omega); \\
(1 - \alpha\omega^2,\ \alpha\omega - 1,\ \omega^2 - \omega), & (1 - \alpha\omega,\ \alpha\omega^2 - 1,\ \omega - \omega^2).
\end{array}$$

The six points just determined together with I_i may be arranged in the scheme

$$\begin{array}{ccc}
(0, 1, -1), & (\omega^2 - \omega,\ 1 - \alpha\omega^2,\ \alpha\omega - 1), & (\omega - \omega^2,\ 1 - \alpha\omega,\ \alpha\omega^2 - 1), \\
(\alpha\omega^2 - 1,\ \omega - \omega^2,\ \alpha\omega - 1), & (-1, 0, 1), & (\alpha\omega - 1,\ \omega^2 - \omega,\ 1 - \alpha\omega^2), \\
(1 - \alpha\omega^2,\ \alpha\omega - 1,\ \omega^2 - \omega), & (1 - \alpha\omega,\ \alpha\omega^2 - 1,\ \omega - \omega^2), & (1, -1, 0),
\end{array}$$

whose rows, columns, right and left hand diagonals satisfy the conditions of collinearity imposed on the nine points of inflexion of a cubic. Then from the scheme, for every value of α,

$$(x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z) \\ + \lambda(\alpha x + y + z)(x + \alpha y + z)(x + y + \alpha z) = 0$$

can be read off as the equation of the pencil of cubics with inflexions at the nine points.

For this pencil the lines

$$\alpha x + y + z = 0, \qquad x + \alpha y + z = 0, \qquad x + y + \alpha z = 0$$

are the sides of the real inflexional triangle; and Σ is the point common to the three real harmonic polars h_i. Hence the result may be stated in the theorem:

The six imaginary points of inflexion of a real cubic are the projections, through the point common to the three real harmonic polars, of the two points equianharmonic to the three real inflexions, upon the sides of the real inflexional triangle.

4. **Generalization.** The value of α depends upon the choice of the line through one of the points I, hence the equation

$$(x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z) \\ + \lambda(\alpha x + y + z)(x + \alpha y + z)(x + y + \alpha z) = 0,$$

depending upon two variable parameters, accounts for a two-fold infinite system of cubics—a syzygetic pencil for each of the single infinity of choices of the line with a given Σ. The double infinity of choices of Σ accounts for the fourfold infinite system of cubics in a plane with the same three real points of inflexion.

As Σ varies in position in the plane, the projections through it of the two equianharmonic points define the totality of imaginary points on the lines of the three real pencils

$$\alpha x + y + z = 0, \qquad x + \alpha y + z = 0, \qquad x + y + \alpha z = 0;$$

that is, every one of these points belongs to at least one inflexional group which includes the three given real points. On the other hand, an imaginary point on a real line not included in the three pencils cannot belong to such an inflexional group, and hence the impossibility of an arbitrary choice of an imaginary point of inflexion for a real cubic when the three real points of inflexion are fixed.

Thus is developed the following theorem:

The imaginary points of inflexion of the fourfold infinite system of real cubics in a plane, with three given real points of inflexion, form the totality of imaginary points on the three pencils of real lines through the three fixed inflexions; and group themselves into ∞^3 sets of six points, two points on one line of each pencil, such that each set of six together with the three fixed real points form an inflexional group.*

* The first half of this theorem was also proved in a former paper by the writer.

Two special cases arising when the arbitrarily chosen line is taken (1) through Σ, giving rational cubics with a conjugate point, and (2) coincident with the line through the three points I, giving degenerate cubics, have been considered in another connection in an earlier paper.

Since three collinear points do not determine a plane, it may further be noted that Σ may be taken as any real point in three-dimensional space, and the theorem extended accordingly.

5. **The critic centers.** It has been noted that the removal of the restriction that Σ be a fixed point gives the system of cubics in the plane two more degrees of freedom. A fixed Σ is a critic center (vertex of an inflexional triangle) for every cubic of the doubly infinite system. This suggests that, with no other restriction, four critic centers chosen arbitrarily in the plane may impose eight conditions and hence determine a singly infinite system of cubics.

Suppose the four points $(1, \pm 1, \pm 1)$ to be critic centers for a real cubic. Since the points are all real, three must be taken as vertices of the real inflexional triangle, say $(1, 1, -1)$, $(-1, 1, 1)$, $(1, -1, 1)$. The cubic consisting of the three sides of the triangle is

$$(y + z)(z + x)(x + y) = 0;$$

and the polar line of the fourth point $(1, 1, 1)$ with respect to this cubic is $x + y + z = 0$. On this line $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$ are the two, and the only two, points whose polar lines pass through $(1, 1, 1)$. Hence under the hypothesis $(1, 1, 1)$, $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$ are the vertices of a second syzygetic triangle which forms the cubic

$$(x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z) = 0,$$

and

$$(x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z) + \lambda(y + z)(z + x)(x + y) = 0$$

is a pencil of cubics with the four chosen points $(1, \pm 1, \pm 1)$ as critic centers. This is identical with the equation of the preceding section when α is equal to zero; hence the hypothesis holds, that is, four real points may be arbitrarily chosen in a plane as critic centers for a cubic. From among the four points there are four choices of three, and any such three may be taken as the vertices of the real inflexional triangle. This gives the theorem:

Four real points chosen arbitrarily in a plane as critic centers for a real cubic determine a syzygetic pencil of cubics as one of four.

The nine points of inflexion for the pencil

$$(x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z) + \lambda(y + z)(z + x)(x + y) = 0$$

are

$$\begin{array}{lll} (0,\ 1,\ -1), & (\omega^2-\omega,\ 1,\ -1), & (\omega-\omega^2,\ 1,\ -1), \\ (-1,\ \omega-\omega^2,\ 1), & (-1,\ 0,\ 1), & (-1,\ \omega^2-\omega,\ 1), \\ (1,\ -1,\ \omega^2-\omega), & (1,\ -1,\ \omega-\omega^2), & (1,\ -1,\ 0), \end{array}$$

which define the sides of the two other inflexional triangles and consequently the remaining six critic centers as

$$\begin{array}{ll} (2\omega-1,\ 1,\ 1), & (2\omega^2-1,\ 1,\ 1); \\ (1,\ 2\omega-1,\ 1), & (1,\ 2\omega^2-1,\ 1); \\ (1,\ 1,\ 2\omega-1), & (1,\ 1,\ 2\omega^2-1). \end{array}$$

The six points just determined are the intersections of

$$y - z = 0, \qquad z - x = 0, \qquad x - y = 0,$$

by the lines joining $(1, 1, -1)$, $(-1, 1, 1)$, $(1, -1, 1)$ to the two points $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$; that is, they are the projections on the three harmonic polars, through the vertices of the real inflexional triangle, of the two points equianharmonic to the three real inflexions.

Then, in the figure, the twelve critic centers are V_1; V_2; V_3; Σ; the two points equianharmonic to I_1, I_2, I_3; and the projections upon h_i, through V_i, of the two equianharmonic points.

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